UNCLASSIFIED

NRL Report 8280

Compression of Ephemerides by Discrete Chebyshev Approximations

ANDRE DEPRIT

Department of Mathematical Sciences
University of Cincinnati, Ohio

HENRY PICKARD

Technical Staff
Space Sciences Division

WALTER POPLARCHEK

University College
University of Cincinnati, Ohio

January 4, 1979

NAVAL RESEARCH LABORATORY
Washington, D.C.

Approved for public release; distribution unlimited.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE — READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER NRL Report 8280	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) COMPRESSION OF EPHEMERIDES BY DISCRETE CHEBYSHEV APPROXIMATIONS		5. TYPE OF REPORT & PERIOD COVERED Final report on one phase of a continuing NRL problem 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Andre Deprit*, Henry Pickard, and Walter Poplarchek†		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Research Laboratory Washington, D.C. 20375		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 611539-14 D03-06.101 RR014-02 RR014-02-41
11. CONTROLLING OFFICE NAME AND ADDRESS		12. REPORT DATE January 4, 1979 13. NUMBER OF PAGES 22
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

Material to be presented at the 1978 AIAA/AAS Astrodynamics Conference.
Material to be published in the *Journal of the Institute of Navigation.*
*Present address: Department of Mathematical Sciences, University of Cincinnati, Ohio
†Present address: University College, University of Cincinnati, Ohio

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Best approximations	Compression of ephemerides
Chebyshev approximations	Polynomial approximation
Chebyshev polynomials	Polynomial interpolation

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Polynomial representations of astronomical ephemerides are usually derived from discrete least-squares approximations. Ideally, to ensure a uniform distribution of the error, one should aim at a continuous Chebyshev approximation. This is feasible when the ephemeris is generated from a literal (analytical or semianalytical) development. But a discrete Chebyshev approximation is a realistic compromise. Application to the moon and geosynchronous satellites has given good results. On the whole, long ranges (several times the sidereal period) may be covered by polynomials of degree 30 to 50 with a moderate error. A low-degree approximation over half the period usually delivers a high

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE
S/N 0102-014-6601

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

UNCLASSIFIED

20. Abstract (Continued)

accuracy. Gibbs' phenomena, i.e. rapid oscillations of increasing amplitudes in the error curve at both ends of the approximation interval, are of course absent, contrary to what usually happens in a least-squares approximation.

UNCLASSIFIED

CONTENTS

UNCLASSIFIED

COMPRESSION OF EPHEMERIDES BY DISCRETE CHEBYSHEV APPROXIMATIONS

INTRODUCTION

There are many users of satellite ephemerides. Users of data from a satellite such as LANDSAT need the satellite's position to correlate the data with a specific location on the earth. The ephemerides of navigation satellites are stored onboard and transmitted to users. Ground stations need ephemerides for antenna pointing. In orbit-prediction programs, the ephemerides of the planets and moon are needed to compute their effects on satellite orbits.

An ephemeris can be provided in several ways. It can be in tabular form, in which the user interpolates between points to obtain positions. Orbital elements (such as NAVSPASUR 1- or 5-card element sets) can be provided from which the ephemeris is reconstructed by using a Brouwer Theory orbit generator. Another method is to represent the ephemeris using polynomial approximation. In this method, the user is supplied with the degree of the approximating polynomial and the coefficients needed to construct the polynomial.

This program began from the desire to develop an efficient method for storing and processing satellite ephemerides onboard satellites. Participation of the senior author in the National Bureau of Standards experiment of broadcasting time from geosynchronous satellites [1-3] generated the problem of loading astronomical ephemerides in the memory of microprocessors and microcomputers. The specific goals are

1. To compress the ephemerides, thereby reducing the occupancy in core and the duration of loading into core,

2. To cover as wide a time span as the recipient equipment will admit, in order to confer it with maximum autonomy,

3. To guarantee as much accuracy as the real-time operations will require,

4. To make the processing of the compressed ephemerides as fast as possible.

Polynomial approximation with Chebyshev polynomials can achieve all of these goals.

APPROXIMATION TECHNIQUES

Interpolation Polynomials

Polynomial interpolation may help somewhat in compressing conventionally tabulated almanacs. Newhall [4] reports how an interpolation in Chebyshev polynomials helped in

Manuscript submitted September 8, 1978.

making the files of the JPL Development Ephemeris Number 96 [5] five times shorter than their homolog in the JPL Ephemeris Number 69 [6]. However, for an ephemeris produced by a numerical integrator of order n and step Δt, interpolation polynomials are valid over intervals equal to $n\Delta t$. We are interested in polynomials extending over 10 to 40 times that interval. From a predictor-corrector of order 10 at the step of 500 s applied to 24-h satellite, the extraction of a polynomial approximation valid for 2, 3, or even 5 days is proposed.

Least-Squares Approximation

Forsaking interpolation polynomials, we turn to approximation polynomials. Let $f(\tau)$ be one of the ephemeris elements. Define the error $y(\tau)$ of a finite expansion by forming the difference

$$y(\tau) = f(\tau) - \sum_{0 < j < m} c_j p_j(\tau)$$

where $p_0, p_1, \ldots, p_n$ are prescribed polynomials while the coefficients $c_0, c_1, \ldots, c_n$ are to be determined. Instead of forcing $y(\tau)$ to vanish at prescribed points (as an interpolation requires), the *average error* is obtained by integrating over the finite range $\alpha \leqslant \tau \leqslant \beta$ in which the function $f(\tau)$ shall be represented.

Conventional astronomers usually define the average error by the definite integral

$$\bar{y}^2 = \frac{1}{\beta - \alpha} \int_\alpha^\beta y^2(\tau)\, d\tau$$

and determine the coefficients c_j of the expansion by the prescription that $\bar{y}^2$ be as small as possible. Least-squares approximations always have a definite solution since an essentially positive quantity always assumes a definite absolute minimum.

The problem is greatly facilitated if the base polynomials satisfy orthogonality conditions of the kind

$$\int_\alpha^\beta P_j(\tau)\, P_k(\tau) = 0 \qquad \text{for } j \neq k.$$

Chebyshev polynomials are orthogonal.

Considerable effort has been spent in the least-squares approximation of planetary ephemerides in series of Chebyshev polynomials. The primary example is Carpenter's theory of the five outer planets [7]: following the method outlined by Clenshaw [8,9], Carpenter develops a least-squares approximation of the right-hand members of the planetary equations, then integrates formally the resulting Chebyshev series. The approximation is iterative: starting with polynomials representing uncoupled elliptic Keplerian motions, it ends up with series of several hundred terms representing the deviations of the orbits of Jupiter, Saturn, Uranus, Neptune, and Pluto from Keplerian motions in the interval from 1800 through 2000. Comparison with a numerical integration [10] from 1653 through 2060 produced displacements less than 0."01 after the third iteration and less than 0."001 after the fourth.

UNCLASSIFIED

Series of several hundred terms are not practical. Least-squares approximations of astronomical ephemerides in Chebyshev polynomials are currently derived from conventionally tabulated ephemerides. The orbit generator Goddard Trajectory Determination System (GTDS) for satellites of the earth or the moon takes its almanacs for the sun, the moon, the sidereal time, and its corrections due to precession and nutation from Chebyshev series of degrees which vary between 10 and 20. These series are valid over time intervals of 20 to 30 days and are constructed from either the JPL files or the Improved Lunar Ephemerides [11].

Techniques of this kind have been in use in producing ephemerides which serve to automatically drive antennas and signal transmissions [12]. They have been transposed to the problem of tracking specific features at the moon's surface [13,14]. With the advent of programmable pocket calculators available at the drugstore counter, the U.S. Naval Observatory is now experimenting with a new type of almanac where the major tables of the American Ephemeris and Nautical Almanac are compressed into standardized Chebyshev series [15].

Chebyshev Approximations

The disadvantage of least-squares approximations is usually the absence of an estimate of the error. The difference between the function and its polynomial approximation in least-squares oscillates with nonuniform magnitude; normally minimal in the middle of the range, the amplitude may greatly increase toward both ends of the interval. A uniform approximation produces an error whose absolute maxima and minima are equal in absolute value over the entire interval. With a uniform representation, there is no need to anticipate a "transient" regime at the beginning of the range wherein the maximum possible error decreases, and a "degradation" regime at the end of the range where one should expect the maximum error to start increasing again in absolute value.

The topology subjacent to the Chebyshev approximation is the metric

$$y^* = \max_{-1 \leqslant \tau \leqslant +1} |y(\tau)|.$$

The purpose of the approximation algorithm is to find the coefficients c_j for which y^* would be minimum. The method is based on a conjecture of Chebyshev proved by Borel [16]. If $g(x)$ is a function continuous in the closed interval (a, b), then there exists a unique polynomial of best approximation of given degree n. The error of this approximation reaches its extreme value at $n + 2$ points at least (there may be more) with alternating signs at these $n + 2$ points.

Stiefel [17] has set up an iterative method to yield a best approximation in a finite number of steps. Assume that a sequence (τ_i) of $n + 2$ points has been selected in the interval and that a polynomial series

$$\phi(\tau) = \sum_{0 \leqslant j \leqslant n} c_j T_j(\tau)$$

has been obtained such that the errors

$$y_i = f(\tau_i) - \phi(\tau_i) \qquad 1 \leqslant i \leqslant n + 2$$

have the same absolute value, say y^0. Consider the maximum y^* over the whole interval. If $y^* = y^0$, then the polynomial series is the best approximation; otherwise $y^* > y^0$. Then select one of the points τ at which the maximum error is reached, exchange it with one of the points in the reference set, and go back to constructing a *levelled* approximation, i.e., one for which the errors at the reference points are all equal in absolute value, but alternate in sign.

Stiefel's algorithm will confront the programmer with two difficulties: (a) levelled approximations are easy to construct [18], but only if Haar's condition is satisfied at the reference points; (b) locating the extrema of the error functions outside the reference points requires that one is capable of determining accurately the first derivative $f'(\tau)$. If the function f is known in literal form, calculating $f'(\tau)$ usually presents no problem.

A program (in PL/I, IBM Level F) to implement Stiefel's iteration [19,20] has been developed, and it has been applied to those astronomical ephemerides which are derived from theories, namely

1. the corrections due to nutation from Woolard's theory,

2. the coordinates of the sun, Mercury, Venus, and Mars from Newcomb's theory of the inner planets,

3. the coordinates of the Galilean satellites of Jupiter from Sampson's tables.

The results are definitely encouraging. For instance, polynomials of degree 14 are sufficient to cover a sidereal period (about 230 days) of Venus at the accuracy of the American Ephemeris. Similar results are obtained for Mars. The situation is less favorable for Mercury. Our experience with the Galilean satellites suggests that the error which an approximation of given degree generates is determined for its principal part by the average eccentricity of its orbit. This correlation will be analyzed more closely later on.

A case where Haar's condition was violated has not been encountered.

The precision of the first and second derivatives is critical. A straightforward derivation of the semianalytical expressions given by a theory is safe, but a numerical differentiation by finite differences is not [21].

DISCRETE CHEBYSHEV APPROXIMATIONS

A problem arises if the Chebyshev approximation technique described in the previous section is applied to orbits of artificial satellites around the earth. These orbits are usually given in the form of tables at equidistant times because they are derived from a numerical integration and a literal development. No reliance can be made on the Steifel-Remez algorithm to compress such ephemerides. The construction proposed by Golub [22] would not produce with enough accuracy the dates at which Remez's ripplings occur.

UNCLASSIFIED

The following compromise was found to be successful: in the range $-1 \leqslant \tau \leqslant +1$, select a sequence $(\tau_i)_{0 \leqslant i \leqslant m}$ and solve the overdetermined linear system

$$f(\tau_i) = \sum_{0 \leqslant j \leqslant m} c_j T_j(\tau_i) \qquad (0 \leqslant i \leqslant m,\ n < m)$$

in the sense of Chebyshev, i.e., determine the coefficients (c_j) which minimize the error estimate

$$y_{n,m} = \max_{1 \leqslant i \leqslant m} |f(\tau_i) - \sum_{0 \leqslant j \leqslant n} c_j T_j(\tau_i)|.$$

To accomplish this, a very efficient algorithm established by Barrodale [23,24] can be used. It is basically the simplex algorithm applied to the dual of the linear programming defined by the minimization of the maximum error. At first a simplicial basis is built and a starter is determined to enter a cycle of exchanges which, in a finite number of steps, leads eventually to the optimum sequence $(c_j)_{0 \leqslant j \leqslant n}$.

Barrodale's program is written in the Fortran dialect WATFOR. The authors of this report have modified it to relax the syntactical restrictions imposed by WATFOR, to satisfy the stylistic rules adopted in structured programming, and to insert it in production programs which compress by segments a conventionally tabulated ephemeris stored in a sequential file. The program is available upon request to the office of the Space Systems Division at the U.S. Naval Research Laboratory. Its input dispositions have been developed to process large ephemerides files; this version is available at the Charles Stark Draper Laboratory.

Illustrations

As a test of what discrete Chebyshev approximations can achieve in compressing ephemerides, consider the ephemerides of the moon.

In principle, a continuous Chebyshev approximation can be used. The lunar ephemeris may be calculated from the semianalytical theory known as the Improved Lunar Ephemeris (type $j = 2$). But the multivariate Fourier series contains several hundred terms, and it is not practical that such a large series be evaluated repeatedly not only for the coordinates (as they must be in any case) but also for their first and second derivatives with respect to the time.

Dr. Van Flandern of the U.S. Naval Observatory provides a program which optimizes the tabulation of a lunar ephemeris at equidistant times from the amended series in the Improved Lunar Ephemeris. The primary interest is in testing how wide a range could be covered while maintaining the precision retained in the American Ephemeris and Nautical Almanac. The errors in absolute value predicted by the algorithm are presented in Table 1. The range is expressed in days; the deviations ΔU, ΔV, $\Delta\alpha$, and $\Delta\delta$ in longitude, latitude, right ascension, and declination are expressed in units equal to 10^{-9} radian, while the error Δr is in units of 10^{-9} times the earth radius. The table shows that a full period of the moon may be covered by Chebyshev polynomials of only degree 24 at a precision of 0.″035 in the angles and of 16 m in the distance (comparable to the accuracy maintained by the American Ephemeris and Nautical Almanac).

Table I — Compression of Lunar Ephemeris

Range	Degree	ΔU	ΔV	Δr	$\Delta\alpha$	$\Delta\delta$
7^d	6	332	259	28897	4840	5674
	7	96	56	529	1706	635
	8	40	4	437	401	157
	9	32	3	323	40	7
	10	48	3	316	53	11
	12	45	3	264	46	11
	14	42	3	226	43	10
	16	39	3	198	40	10
	20	36	2	131	37	9
	24	34	2	102	35	9
	28	31	2	90	32	8
14^d	10	256	81	576	630	603
	12	44	7	164	823	806
21^d	14	107	59	3133	6746	2752
	16	36	11	2213	1901	357
	18	28	3	1988	377	132
28^d	18	71	40	2453	5425	1337
	20	39	15	2321	5196	699
	22	38	4	2791	525	136
	24	36	2	2471	171	35
	28	23	2	1274	25	6
	40	33	2	1557	34	8
	44	31	2	1457	32	7
30^d	20	34	44	3304	1317	1019
56^d	24	23157	17245	291736	184845	141263
	36	316	211	4493	2998	2773
	44	83	16	2170	809	109
	50	76	5	1549	94	36

A similar analysis of possible ranges and degrees was made for the satellite SMS-B of NOAA. The orbit was produced by GTDS, the geocentric coordinates being filed at the rate of one state vector every 15 min. A shorter filing step would have improved the accuracy of the interpolation in the data set. The best residuals predicted by Barrodale's algorithm turned out to be comparable to the interpolation errors. Table 2 suggests the type of performance that one may expect from the compressed ephemerides. At the National Bureau of Standards, in the present stage of development, accuracy requirements are moderate without being acute, but it is critical to secure as long an autonomy of the equipment installed at Wallops Island as is compatible with a time broadcast to better than 10 μs to general users. In that context, it is quite encouraging to read in Table 2 that several days of

UNCLASSIFIED

Table 2 — Compression of Ephemerides
for the 24-h Satellite SMS-B

Range	Degree	Δr	ΔU	ΔV
$0^d.25$	4	575	43	310
	8	3	3	4
	12	2	2	1
	16	0	0	0
$0^d.50$	8	228	9	5
	12	5	3	4
	16	4	3	3
	20	2	2	2
$0^d.75$	8	860	64	43
	12	9	6	6
	16	5	4	4
	20	4	3	4
$1^d.$	12	83	11	5
	16	5	5	7
	20	5	4	5
	24	5	4	4
$2^d.$	16	3165	401	19
	20	347	56	9
	24	55	12	6
	28	15	5	6
$3^d.$	20	11676	2019	110
	24	2424	502	37
	28	617	124	13
	32	226	31	7
$4^d.$	24	36057	6433	389
	28	5984	1068	67
	32	2974	529	39
	36	968	201	18
$5^d.$	28	67211	13197	798
	32	16257	2718	186
	36	4653	780	55
	40	3345	585	42

ephemerides may be compressed in Chebyshev series of reasonable degree. There, the residuals between data from integration and values from minmax approximations are in cm for the distance r and in 10^{-7} degrees for longitude U and latitude V.

In Fig. 1, the error in distance is plotted for an approximation of degree 20 over two days. The curve is typical of a discrete Chebyshev approximation. The rippling character is well in evidence; maxima and minima are not quite at the same height, which shows that the Chebyshev series narrowly misses being the best approximation of degree 20 for the distance over two days.

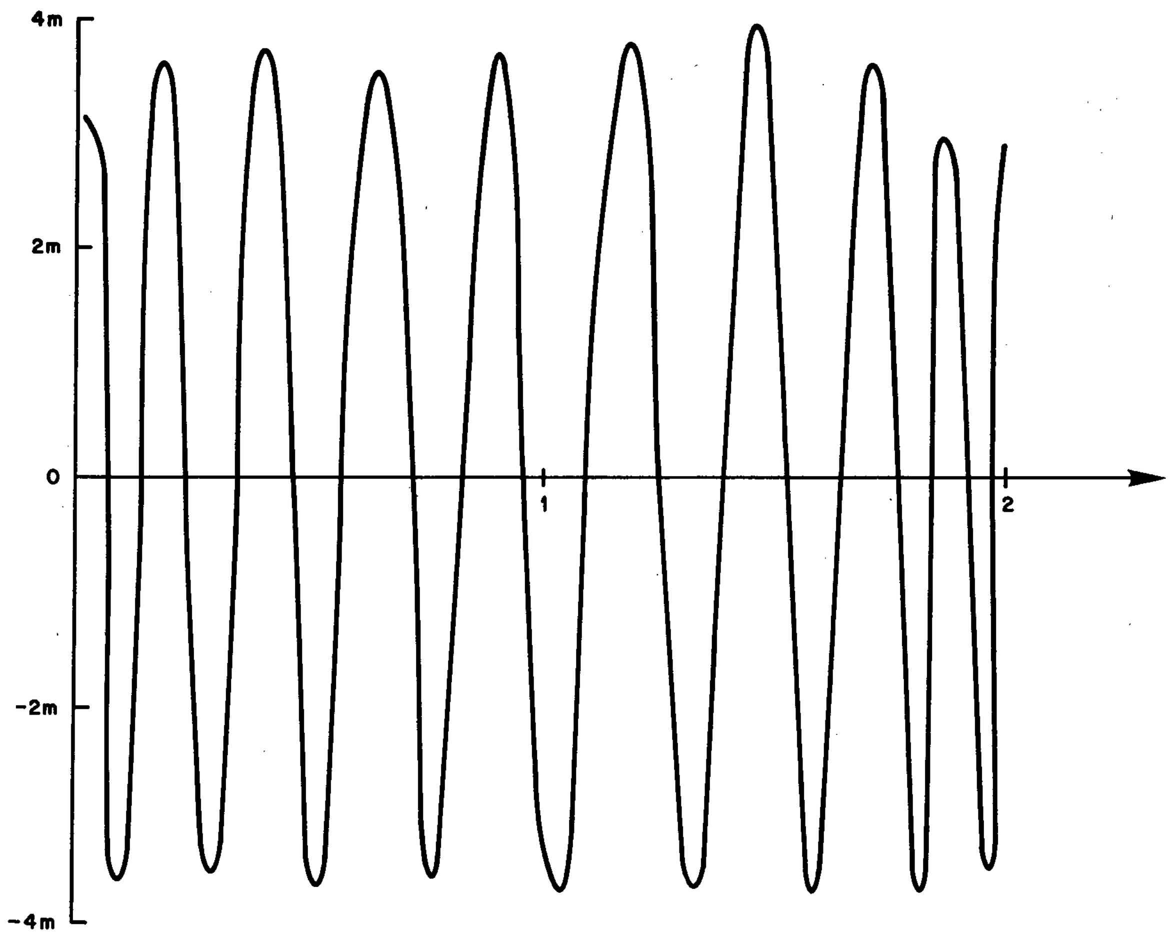


Fig. 1 — Error curve for a discrete Chebyshev approximation

UNCLASSIFIED

Selection of the Reference Points

High-frequency oscillations of the truncated series around the true function are always present. With a least-squares approximation they are very noticeable and interfere most adversely with an efficient polynomial synthesis. The error plot in Corio [25] is a striking illustration of the Gibbs phenomenon: for a 24-h satellite, an error of less than 1 m around noon causes wild oscillations of 10 km on both sides of midnight. Corio's figure is in sharp contrast with the error curve in Fig. 1.

Fejer's rule of taking the arithmetic mean

$$\frac{1}{n+1} \sum_{0 \leqslant j \leqslant n} S_j(\tau)$$

where

$$S_j = \sum_{0 \leqslant k \leqslant j} c_k T_k \qquad (0 \leqslant j \leqslant n)$$

eliminates completely the Gibbs phenomenon [26]. Yet it must be said that the method approaches the limit value very slowly. The method of smoothing the coefficients of the approximation [27,28] is a somewhat simpler process; it does not eliminate the oscillations but cuts down their amplitudes.

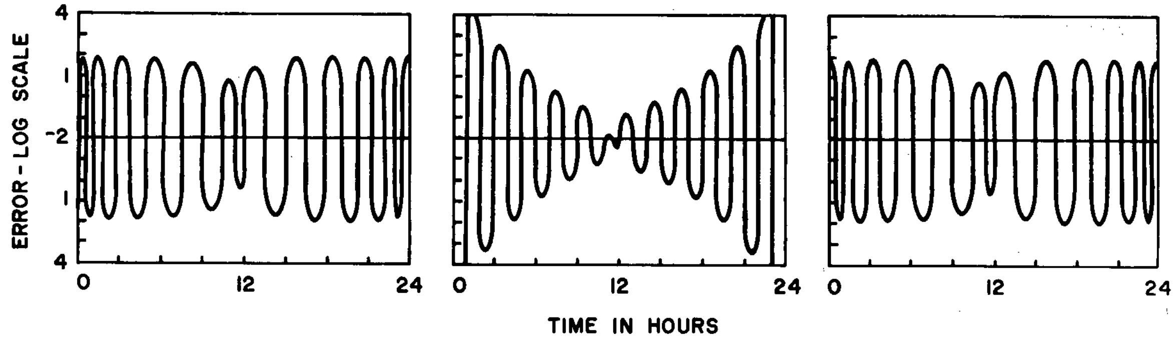
On the whole, however, much can be accomplished toward damping the Gibbs oscillations by securing a nonuniform distribution of the approximation points which crowds the points near the two end points $\tau = \pm 1$ of the range. The effectiveness of this method is illustrated in Fig. 2, (a) and (b). Plotted here are error curves corresponding to polynomial approximations of a 12-h satellite of eccentricity 0.01 and inclination 63.4° over two periods. Twenty-five points were used to generate the approximations of Fig. 2(a) and sixty points were used for those of Fig. 2(b). The polynomial approximations corresponding to the three plots of each figure were generated as follows:

Plot 1: Reference points were located at the zeros of the Chebyshev polynomial of order 25 (Fig. 2(a)) and 60 (Fig. 2(b)). The Barrodale algorithm was used.

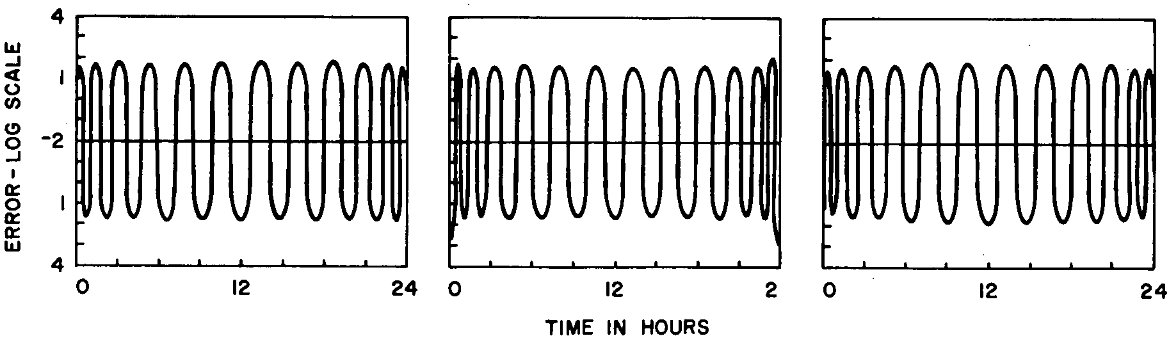
Plot 2: Reference points were uniformly distributed. The Barrodale algorithm was used.

Plot 3: Reference points were located as in Plot 1, but least-squares fitting was used.

The contrast in results between a uniform and a nonuniform distribution of reference points is most striking when only 25 points are used, but the effect is still significant for 60 points. In Fig. 2(a), the accuracy of the approximation corresponding to a uniform distribution deteriorates steadily and drastically from the midpoint to the end points of the interval. The case for the uniform distribution is not nearly so bad if 60 points are used, but errors at the very end points still extend by a factor of 10 above those corresponding to nonuniformly distributed points. The third plot in each figure is included for comparison between the least-squares fitting and the Barrodale algorithm. The results are essentially the same for these two methods. The error curve of a least-squares fit for uniformly distributed points is very similar to the second plot and has not been included in Fig. 2.



(a) Twenty-five reference points



(b) Sixty reference points

Fig. 2 — Error curves for the 24th-degree fit

Only when a large number of reference points is used can the results of a uniform distribution match the results of a nonuniform distribution. Figure 3 shows how well the Barrodale maximum fitting error estimates the true maximum error over the entire interval as a function of the number of reference points used. The solid curves indicate true errors and the dashed curves indicate errors given by the Barrodale routine. Curves 1 correspond to the case where the reference points are taken at the zeros of the Chebyshev polynomial of order given by the x-axis. Curves 2 correspond to the case where the reference points are uniformly distributed. The 12-h satellite orbit used previously was also used in generating these plots. For these cases, it takes 60 to 100 uniformly distributed points to match the accuracy given by 25 nonuniformly generated points. Furthermore, although the predicted error is less for uniformly distributed points than for nonuniformly distributed points, the true error is greater for the uniform distribution, even if a large number of points is used. Thus, while the true error is underestimated in both cases, reliable prediction is at least possible even with only 25 reference points when they are concentrated near the end points of the interval.

With proper crowding of the reference points toward the end points, the error oscillates with the same order of magnitude throughout the range. The error profile for a least-squares approximation is now similar to that of a discrete Chebyshev approximation. The latter, however, would still be preferable since it produces an estimate of the least maximum

UNCLASSIFIED

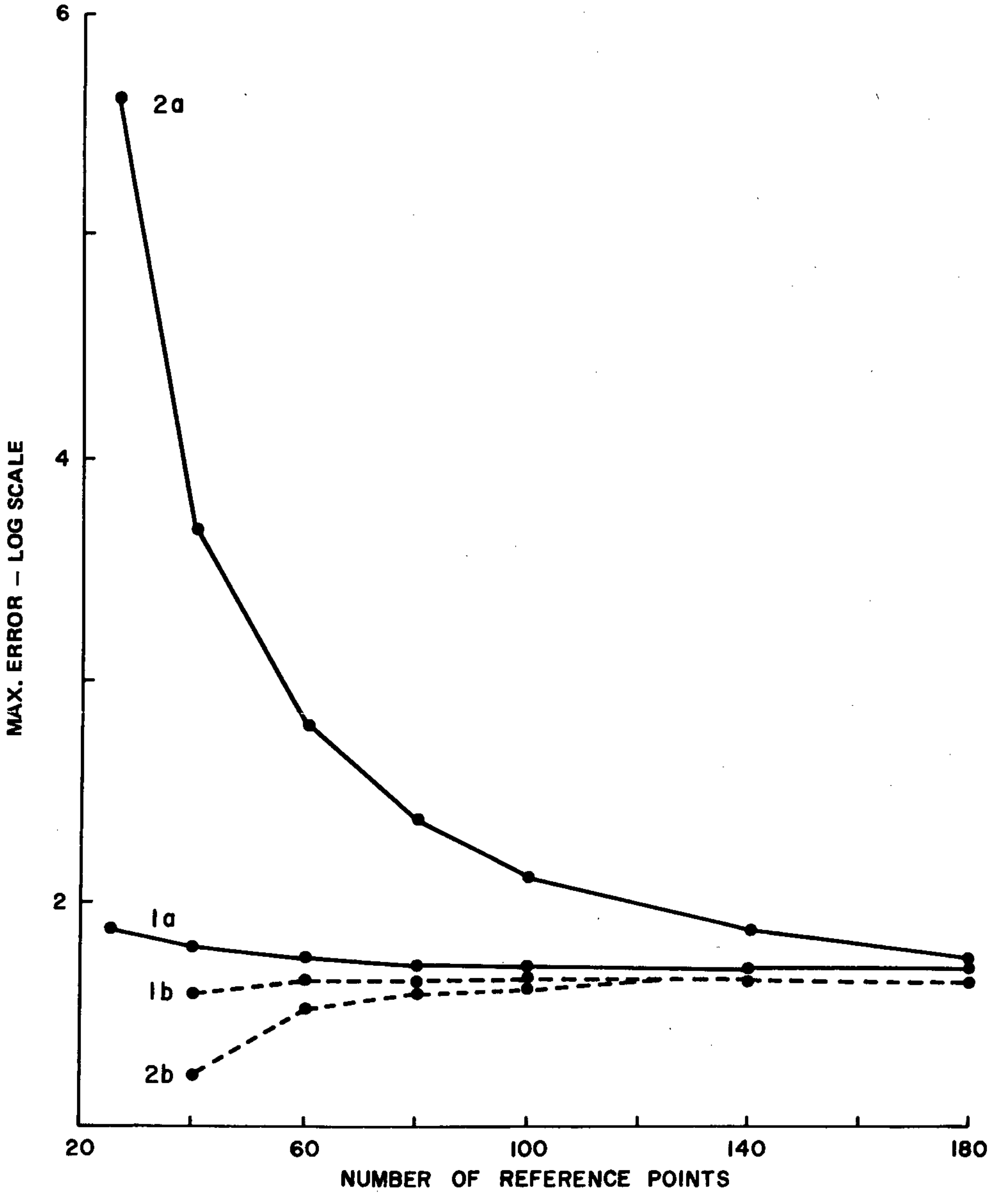


Fig. 3 — Comparison of polynomial approximations using uniform and nonuniform reference-point distributions for several numbers of reference points

error that may be reached for a given degree. This specific information is critical in the design of a microprocessor or in the programming of a minicomputer to process compressed ephemerides.

Convergence and Eccentricity

To get an idea of how well a Chebyshev series can approximate orbit functions, the Barrodale algorithm has been used to generate approximating polynomials for elliptic orbits of different eccentricities. The following discussion and accompanying tables give a rough estimate of what order is needed to approximate a given Keplerian element to within a specified error.

Tables 3-6 show results from a 12-h orbit of various eccentricities. The inclination was 63.4° and the argument of perigee was on the equator. The approximating polynomials were generated by using 60 reference points over the interval of consideration; the points were chosen at the zeros of the Chebyshev polynomial of degree 60. The maximum error in fitting these 60 points was given by the Barrodale algorithm, and this error was used initially to determine the lowest degree needed to fit the entire interval to within the specified error. The resulting approximating polynomial was then compared to the actual function at 500 points over the interval to determine the true maximum error. The maximum error given by the Barrodale algorithm is always optimistic to some degree, and in some cases this causes the minimum degree to be underestimated. Experience so far indicates that for small to moderate eccentricities the maximum error at the 60 points is optimistic by no more than about 10 to 20 percent. Thus, only when this error falls very close to the maximum error specified is it necessary to increase the order to ensure the desired accuracy over the entire interval. It is stressed, however, that in general, this holds only when the reference points are chosen with a concentration near the end points of the interval. If a uniform distribution of points is used, it takes considerably more points to achieve the same accuracy. In the tables, an asterisk indicates that the degree needed is greater than 59.

Tables 3-6 show results for the radius component and for the x-component in an Earth-fixed cartesian coordinate system. Results for the y- and z-components are analogous to those for x. Problems were encountered in fitting latitude and longitude, and these are discussed separately.

Experience suggests the following conclusions in regard to fitting the cartesian components of elliptic orbits. Large eccentricity causes the greatest difficulty in fitting orbit functions. The degree needed to achieve a specified accuracy becomes sensitive to eccentricity when it exceeds 0.01 and extremely sensitive when it gets above 0.1. Not only does the necessary degree become large, but the accuracy improves very slowly with increasing degree. As Tables 3 and 4 show, the increase in degree needed to improve the accuracy by 10^{-4} is only 4 for $e = 0.001$, but jumps to 20 for $e = 0.5$ and to 40 for $e = 0.75$. Extending the time interval of consideration also has a great effect on the degree needed to fit, but this is not a problem inherent to the particular coordinate of interest. The pertinent question in this case is whether or not it is advisable to fit one large-degree polynomial rather than two smaller degree polynomials to two or more periods of an orbit. Tables 5 and 6 show that for small eccentricities, two periods can be fitted with less than double the degree needed to fit one period. For $e = 0.5$, however, more than triple the degree is needed for two periods. The general conclusion appears to be that the cartesian coordinates of orbits can be fitted very well if the eccentricity is small, but with difficulty when the eccentricity is large.

The discussion above still holds when one tries to fit either latitude or longitude, except that the inclination has a very definite effect on the results. In fact, the longitude for a 63.4° inclined orbit is already very difficult to fit, so that results in Table 7 are for a 10° inclined orbit. This sensitivity to inclination is attributed to the very rapid change of longitude at high latitudes. Aside from the sensitivity to inclination, fitting the longitude is generally more difficult than fitting the cartesian components and, therefore, results are shown for only one period. Fitting the longitude for orbits of inclination greater than 45°, even for one period, is deemed impractical. When it is desired to fit the longitude of an inclined orbit, the procedure should be to first rotate the coordinate system so that the x-y plane coincides with the orbital plane.

UNCLASSIFIED

Table 3 — Minimum Degree Necessary to Fit Radius Components of Elliptic Orbits Over One Period

Max. Error	Degree					
	0.0†	0.001†	0.01†	0.1†	0.5†	0.75†
10 km	0	4	4	6	12	28
1 km	0	4	6	8	18	30
100 m	0	6	8	12	24	42
10 m	0	8	10	12	26	48
1 m	0	8	12	16	34	*

*Indicates that the degree needed is greater than 59.
†Eccentricity

Table 4 — Minimum Degree Necessary to Fit X-Component of Elliptic Orbits Over One Period

Max. Error	Degree					
	0.0†	0.001†	0.01†	0.1†	0.5†	0.75†
10 km	9	9	9	11	15	18
1 km	11	11	11	13	17	31
100 m	13	13	13	15	25	42
10 m	15	13	15	17	31	49
1 m	15	15	15	19	35	*

*Indicates that the degree needed is greater than 59.
†Eccentricity

Table 5 — Minimum Degree Necessary to Fit Radius Component of Elliptic Orbits Over Two Periods

Max. Error	Degree					
	0.0†	0.001†	0.01†	0.1†	0.5†	0.75†
10 km	0	6	8	16	59	*
1 km	0	8	12	22	*	*
100 m	0	10	14	28	*	*
10 m	0	12	18	36	*	*
1 m	0	14	22	42	*	*

*Indicates that the degree needed is greater than 59.
†Eccentricity

Table 6 — Minimum Degree Necessary to Fit X-Component of Elliptic Orbits Over Two Periods

Max. Error	Degree					
	0.0†	0.001†	0.01†	0.1†	0.5†	0.75†
10 km	16	16	18	24	*	*
1 km	18	18	20	32	*	*
100 m	20	20	24	38	*	*
10 m	22	24	28	45	*	*
1 m	22	26	30	52	*	*

*Indicates that the degree is greater than 59.
†Eccentricity

Table 7 — Minimum Degree Necessary to Fit the Longitude of Elliptic Orbits Over One Period

Max. Error (Radians x 10^{-6})	Degree						Max. Error (Sec. of Arc)
	0.0†	0.001†	0.01†	0.1†	0.5†	0.75†	
1000.0	7	5	7	7	13	27	206.0
100.0	9	9	9	9	19	35	20.6
10.0	11	13	13	13	23	45	2.06
1.0	15	15	15	15	29	55	0.206
0.1	17	17	17	17	35	*	0.0206

*Indicates that the degree needed is greater than 59.
†Eccentricity

DERIVATIVE OF A CHEBYSHEV SERIES

Clenshaw has proposed a recursive algorithm to evaluate expressions of the type

$$f(\tau) = \sum_{0 \leqslant j \leqslant n} c_j T_j(\tau)$$

in a straightforward manner, i.e. without expanding the Chebyshev polynomials. It is definitely of interest to supplement the procedure with one to calculate $f'(\tau)$ immediately from $f(\tau)$ without having to execute and expand the derivatives. This means that in cases where the time rates of the ephemeris coordinates would be of use, one would not have to generate and transmit polynomial approximations for the derivatives. The procedure is based on the following lemma [29]:

UNCLASSIFIED

Let (p_j), (α_j), and (β_j) (for $0 \leqslant j$) be three sequences of functions such that

$$p + \alpha_0 p_0 = 0$$

$$p_{n+2} + \alpha_{n+1} p_{n+1} + \beta_n p_n = 0 \qquad (n \geqslant 0).$$

Given the sum

$$f(\tau) = \sum_{0 \leqslant j \leqslant n} c_j p_j(\tau),$$

construct the sequence $(b_j)_{0 \leqslant j \leqslant n+2}$ by recurrence starting with

$$b_{n+2} = 0, \qquad b_{n+1} = 0$$

and looping through

$$b_j + \alpha_j(\tau)\, b_{j+1} + \beta_j(\tau)\, b_{j+2} = c_j$$

for j decreasing from n to 0. Then the value of f at τ is the sum

$$f(\tau) = (b_0 + \alpha_0(\tau)\, b_1)\, p_0(\tau) + b_1 p_1(\tau).$$

When p_j is the Chebyshev polynomial of degree j, Clenshaw's fundamental recursive identities are satisfied by

$$\alpha_0 = -\tau, \qquad \alpha_j = -2\tau \qquad \text{for } j \geqslant 1,$$

$$\beta_j = 1 \qquad \text{for } j \geqslant 0.$$

Therefore, the series (b_j) is given by the recursive equalities

$$b_{n+2}\,,\, b_{n+1} = 0$$

$$b_j = 2\tau\, b_{j+1} - b_{j+2} + c_j \qquad \text{for } n \geqslant j \geqslant 1,$$

and it yields the final result

$$f(\tau) = b_1 \tau - b_2 + c_0.$$

In order to apply Clenshaw's lemma to the evaluation of $f'(\tau)$, introduce the variable θ such that

$$\tau = \cos\theta$$

and recall that, in terms of θ,

$$T_j = \cos j\theta. \qquad (j \geqslant 0).$$

Now putting

$$U_j(\tau) = \sin (j + 1)\theta/\sin\theta,$$

it is found that

$$T_j'(\tau) = j\,U_{j-1} \qquad (j \geqslant 1)$$

and that

$$f'(j) = \sum_{0 \leqslant j \leqslant n} (j + 1)\, c_{j+1} U_j .$$

From the obvious identity

$$U_j(\tau) = 2\tau\, U_{j-1}(\tau) + U_{j-2} = 0,$$

it is found that the functions U_j verify the conditions of Clenshaw's lemma for

$$\alpha_j = -2\tau \text{ and } \beta_j = 1 \text{ for } j \geqslant 0.$$

Hence the sequence

$$b_{n+1}, b_n = 0,$$

$$b_j = 2\tau b_{j-1} - b_{j+2} + (j + 1)c_j \qquad (n\text{-}1 \geqslant j \geqslant 0)$$

leads to the evaluation

$$f'(\tau) = b_0.$$

Note that the above algorithm is shorter and simpler than the one proposed by Broucke [30].

In both procedures, the sequence (b_j) is not properly a one-dimensional array (one does not need to save the intermediate values (b_j)) but a three-cell push-down stack. Thus Clenshaw's procedures are particularly suited to microprocessors or microcomputers designed around the chip of a pocket calculator.

CONCLUSIONS

The problem of transmitting and processing ephemerides in real time has troubled astronomers now for some time. It is generally thought that approximations in Chebyshev polynomials are the solution to the problem.

UNCLASSIFIED

The shorter the range of approximation, the lower the error. But if the degree is allowed to take somewhat large (in excess of 10) values, the range may easily extend over several orbital periods. This contradicts the conclusion drawn by VanDierendonck [31] that approximations by polynomials cannot cover more than a full period of the orbit.

Proper crowding of the data points at both ends of the range damps the Gibbs oscillations at the extremities to the extent that the error curve of a least-squares fit tends to acquire the uniform rippling character of a Chebyshev (also called minmax or "best") approximation. Nevertheless, discrete Chebyshev approximations are favored because they rely on global error to iterate the calculation of the coefficients and thus produce directly an estimate of the maximum error over the range. On the whole, they produce shorter series, i.e., without tails; the range of the coefficients is narrower, and this facilitates the task of economically sizing the bit length to stack the coefficients in the transmission message. On this point though, the reader should be cautioned against believing that the authors advocate a particular set of elements as being the most economical one. It so happens that, either for the 24-h satellites or for the moon, the context in which the series would be used requires spherical coordinates. It is likely that orbital elements rather than coordinates would be more economical: the slow-varying elements would give rise to polynomial approximations of very low degree, and only the fast-varying element would reach the kind of high degrees mentioned previously in Tables 1 and 2. For orbits at low inclinations with small eccentricities, the *ideal* coordinates [32] or a variant thereof called the *equinoctial* elements is suggested[33].

REFERENCES

1. D.W. Hanson and W.F. Hamilton, "Time and Frequency Broadcast Experiments from the ATS-3 Satellite," NBS Technical Note 645 (1973).
2. D.W. Hanson and W.F. Hamilton, "Satellite Broadcasting of WWV Signals," IEEE Trans. Aerosp. Electron. Syst., AES No. 5, p. 562.
3. J.V. Cateora, D.D. Davis and D.W. Hanson, "A Satellite-Controlled Digital Clock," NBS Technical Note 681, (1976).
4. XX Newhall, "Compact Numerical Planetary Ephemerides," Am. Astron. Soc., Div. Dyn. Astron., 7th Annual Meeting, Pasadena, California, Dec. 12, 1975.
5. E.M. Standish, M.S.W. Keesey, and X.X. Newhall, "JPL Development Ephemeris Number 96," NASA Tech. Rep. 32-1603, Jet Propulsion Laboratory, California Institute of Technology, p. 7, (1976).
6. D.A. O'Handley, D.B. Holdridge, W.G. Melbourne, and J.D. Mulholland, "JPL Development Ephemeris Number 69," NASA Tech. Rep. 32-1465, Jet Propulsion Laboratory, California Institute of Technology (1969).
7. L. Carpenter, "Planetary Perturbations in Chebyshev Series," NASA Tech. Note D-3768, NASA Goddard Space Flight Center, Greenbelt, Maryland, 91 p., (1966).
8. C.W. Clenshaw, "The Numerical Solution of Linear Differential Equations in Chebyshev Series," Proc. Cambridge Philos. Soc. 53, 134-149 (1957).

9. C.W. Clenshaw and H.J. Norton, "The Solution of Nonlinear Ordinary Differential Equations in Chebyshev Series," Comput. J. **6**, 88-92 (1963).

10. W.J. Eckert, D. Brouwer, and G.M. Clemence, "Coordinates of the Five Outer Planets 1653-2060," Astronomical Papers prepared for the use of the American Ephemeris and Nautical Almanac, U.S. Government Printing Office, **12**, 327 p., (1951).

11. J.0. Cappellari, C.W. Velez, and A.J. Fuchs, "Mathematical Theory of the Goddard Trajectory Determination System," NASA Rep. X-582-76-77, Goddard Space Flight Center, Greenbelt, Maryland, 3.73-3.80 (1976).

12. C.L. Lawson, "User's Guide to the program LASER 3," JPL Section 391, TM 169, March 1971.

13. J.D. Mulholland, "User's Guide to the program LASER 3," JPL Section 391, TM 171, March 1971.

14. J.D. Mulholland, "Program LASER 7T, User's Guide and Program Description," University of Texas, Dept. Astron., May 1973 (private communication).

15. G.H. Kaplan, L.E. Doggett, and P.K. Seidelmann, "Almanac for Computers, 1977," U.S. Naval Observatory Circular No. 155, Oct. 1976.

16. C. de la Vallée-Poussin, *Lecons sur l'approximation des fonctions d'une variable réelle*, Paris Gauthier-Villars p. 75 (1919).

17. E.L. Stiefel, "Numerical Methods of Tchebycheff Approximation," in *On Numerical Approximation*, R.E. Langer, ed, the University of Wisconsin Press, Madison, 217-232 (1959).

18. E. Stiefel, "Ueber diskrete und lineare Tschebyscheff-Approximationen," Numerische Mathematick **1**, 1-28 (1959).

19. A. Deprit, W. Poplarchek, and A. Deprit-Bartholomé, "Compression of Ephemerides," *Celestial Mechanics*, 11, 53-58 (1975).

20. A. Deprit, "Constructions d'ephemerides condensées," C. R. Ac. Sc. de Paris, 278, 1055-1057 (1974).

21. A.T. Sinclair, Royal Greenwich Observatory, Herstmonceux, Great Britain. A private communication at the XVIth General Assembly of the IAU, Grenoble Aug. 24-Sept. 2 1976.

22. G.H. Golub and L.B. Smith, "Chebyshev Approximation of Continuous Functions by a Chebyshev System of Functions," Comm. ACM, 14, 737-746 (1971).

23. I. Barrodale and C. Phillips, "An Improved Algorithm for Discrete Chebyshev Linear Approximations," Proc. 4th Manitoba Conf. Numer. Math., U. of Manitoba, Winnipeg, 177-190 (1974).

24. I. Barrodale and C. Phillips, "Algorithm 495. Solution of an Overdetermined System of Linear Equations in the Chebyshev Norm," ACM Trans. Math. Software, 1, 264-270 (1975).

25. A.J. Corio, "The Use of Chebyshev Polynomials for Satellite Ephemerides," COMSAT Tech. Rev., **3**, 411-418 (1973).

UNCLASSIFIED

26. L. Fejer, "Untersuchungen über Fouriersche Reihen," Mathematische Annalen, 58, 51-69 (1904).

27. E.T. Whittaker and G.N. Watson, *A Course of Modern Analysis*, Cambridge University Press, 167-168 (1927).

28. C. Lanczos, *Applied Analysis*, Prentice Hall, 225-229 (1956).

29. C.W. Clenshaw, "A Note on the Summation of Chebyshev Series," in *Mathematical Tables and Other Aids to Computation*, 9, 118-120 (1955).

30. R. Broucke, "Ten Subroutines for the Manipulation of Chebyshev Series," Commun. ACM, 16, 294-296 (1973).

31. A.J. VanDierendonck, S.S. Russell, E.R. Kopitske, and M. Birnbaum, "The GPS Navigation Message," AIAA Guidance and Control Conference, San Diego, California, 15 p., Aug. 16-18, 1976.

32. A. Deprit, "Ideal Elements for Perturbed Keplerian Motions," J. Res. NBS, 79B, 1-15 (1975).

33. R.A. Broucke and P.J. Cefola, "On the Equinoctial Orbit Elements," Celestial Mechanics, 5, 303-310 (1972).